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Crystal families and systems in higher dimensions, and geometrical symbols of their point groups. II. Cubic families in five- and *n*-dimensional spaces

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This paper is devoted to the study of the crystal families with cubic symmetries and to the mathematical construction of all their point-symmetry groups. The mono cubic crystal families of *n*-dimensional space (E^n) are defined and a list of these families is given for spaces E^4, E^5, E^6 and E^7 with the Weigel–Phan–Veysseyre (WPV) symbols of their holohedries. The cubic and iso cubic crystal point-symmetry groups of space E^n are also defined together with their properties and their WPV symbols. Some examples of these point groups are given. The 16 point groups of the three isomorphic mono cubic crystal families, the cubic-al family of space E^4 (No. 19), the cube oblique (or cube parallelogram) family of space E^5 (No. XVIII) and the triclinic cubic family of space E^6 (No. 21) are listed. All the WPV symbols of the point-symmetry groups of all the mono cubic crystal families of space E^5, *i.e.* the cube rectangle family (No. XXII), the cube square family (No. XXVI) and the cube hexagon family (No. XXVII), are given together with an explanation of the mathematical construction of these point-symmetry groups. All the di cubic crystal families of spaces E^6, E^7 and E^8 are predicted and the symbols of their holohedries are given. Finally, some tri cubic crystal families of spaces E^9, E^{10} and E^{11} are listed.

1. Introduction

In order to assign a Weigel–Phan–Veysseyre (WPV) symbol (Weigel *et al.*, 1987) to each point group, we follow the same process as in the preceeding paper (Veysseyre *et al.*, 2008), *i.e.* we use two complementary methods:

(1) A computer-analysis method written by H. Veysseyre (Veysseyre & Veysseyre, 2002), which gives for each crystal family the number of point groups of the holohedry and for each point group the elements.

(2) A geometrical study of the family cells.

Owing to these two independent methods, a clear WPV symbol can be given to each point group. Moreover, this symbol allows us to find all the elements of the group and the crystal family to which it belongs.

In this paper, the point groups of the following four crystal families of space E^5 are studied: cube oblique (No. XVIII), cube rectangle (No. XXII), cube square (No. XXVI) and cube hexagon (No. XXVII). These families have in common the following property: their cell is built from a unique cubic subcell, hence the name 'mono cubic' crystal families. This cube belongs to space E^3 generated by axes *x*, *y*, *z*, orthogonal two by two. Over the cube, the family subcells are one of the following four subcells belonging to space E^2, orthogonal to space E^3 and generated by axes *t*, *u*: oblique, rectangle, square or hexagon. The WPV symbols of the holohedries of these families are, respectively, $m\overline{3}m\perp 2$, group of order 48 × 2 = 96, $m\overline{3}m\perp mm$, group of order 48 × 2 × 2 = 192, $m\overline{3}m\perp 4mm$, group of order 48 × 8 = 384 and $m\overline{3}m\perp 6mm$, group of order 48 × 12 = 576. We recall that cube oblique means the two subcells cube and rectangle belong to two orthogonal subspaces of space E^5.

We note that two important papers by Janssen *et al.* (1999, 2002) describe crystallographic notions and in particular symbols for some point groups. These authors often use symbols different from those that we give. We explained how to obtain one symbol from the other in the preceeding paper (Veysseyre *et al.*, 2008). For instance, $432(m1m)$ is equivalent to $\overline{4}3\overline{1}$ (system 19_1). Instead of the symbol $[5].\overline{4}3m\times\overline{1}_4$, we prefer the equivalent symbol $[10].\overline{4}3m$ because $[5].\overline{1}_4$ equals $[10]$.

2. Mono cubic crystal families of spaces E^n

2.1. Definition of the mono cubic crystal families

A crystal family of space E^n is called mono cubic if its cell is built from a unique cubic cell belonging to a subspace E^3 of space E^n and a non-cubic cell belonging to space E^{n-3}

Table 1
Non-cubic crystal families of space E^4 and their holohedries.

This table lists the 21 non-cubic crystal families of space E^4 and their holohedries. [8], [10] and [12] are the short notation for cyclic groups of space E^4 generated, respectively, by the double rotation $8^1 8^3$ by angles $2\pi/8$ and $6\pi/8$, by the double rotation $10^1 10^3$ by angles $2\pi/10$ and $6\pi/10$, and by the double rotation $12^1 12^5$ by angles $2\pi/12$ and $10\pi/12$. *mm* is short for $m\perp m$ and *mmmm* is short for $m\perp m\perp m\perp m$.

Family	Holohedry
Hexaclinic	$\bar{1}_4$
Triclinic-al	$\bar{1}\perp m$
Di obliques	$2\perp 2$
Oblique rectangle	$2\perp mm$
Di rectangles	*mmm*
Square oblique	$4mm\perp 2$
Hexagon oblique	$6mm\perp 2$
Diclinic di squares	44
Diclinic di hexagons	66
Square rectangle	$4mm\perp mm$
Hexagon rectangle	$6mm\perp mm$
Monoclinic di squares	44.2
Monoclinic di hexagons	66.2
Di squares	$4mm\perp 4mm$
Hexagon square	$6mm\perp 4mm$
Di hexagons	$6mm\perp 6mm$
Monoclinic di iso squares	[8].2
Monoclinic di iso hexagons	[12].2
Decagonal	[10].2
Di iso hexagons	[12].2.6*mm*
Rhombotope $\cos\alpha = -1/4$	[10].$\bar{4}3m$

orthogonal to the previous space E^3 (Phan *et al.*, 1988; Veysseyre *et al.*, 1991).

2.1.1. Characteristic property. The holohedry WPV symbol of a mono cubic crystal family has one and only one symbol $m\bar{3}m$, which is the Hermann–Mauguin holohedry symbol of the cubic family of space E^3.

2.1.2. The mono cubic crystal families of spaces E^3 and E^4. In space E^3, there exists only one mono cubic crystal family: the cubic family.

In space E^4, there exist two mono cubic crystal families:

(i) The cubic-al family. Its cell is a right hyperprism with a cube for basis, and it depends on two parameters: the length *a* of the cube side and the height *b* of the side of the hyperprism. The WPV holohedry symbol of this family is $m\bar{3}m\perp m$, group of order $48 \times 2 = 96$.

(ii) The hypercube 4-dim. family obtained when the lengths *a* and *b* are equal. Its cell is limited by eight cubes (Veysseyre *et al.*, 1984; Veysseyre, 1987). The WPV holohedry symbol of this family is $m\bar{3}m$.[8], group of order $48 \times 8 = 384$, [8] being the short notation for a cyclic group generated by the double rotation $8^1 8^3$ by angles $2\pi/8$ and $6\pi/8$.

2.2. Building the mono cubic crystal families of spaces E^5, E^6 and E^7

2.2.1. Introduction. In this section, H_i is the holohedry notation of a non-cubic crystal family of space E^i, orthogonal to space E^3 (the space that contains the cubic cell by definition). The holohedry definition is given in Janssen *et al.* (1999). There exist:

(i) One holohedry H_1 in space E^1, *i.e.* group *m*.

(ii) Four holohedries H_2 in space E^2, *i.e.* groups 2, *mm*, 4*mm*, 6*mm*, the holohedries of the oblique, rectangle, square and hexagon families, respectively.

(iii) Five holohedries H_3 in space E^3, *i.e.* groups $\bar{1}$, $2\perp m$, *mmm*, $4mm\perp m$, $6mm\perp m$, the holohedries of the triclinic, monoclinic, orthorhombic, tetragonal and hexagonal families, respectively.

(iv) Twenty-one holohedries H_4 in space E^4. These groups as well as the crystal families are listed in Table 1. These results have been given by Phan *et al.* (1988).

2.2.2. The mono cubic crystal families of space E^5. In space E^5, there exist six mono cubic crystal families:

(i) Four reducible crystal families, with a group of type $m\bar{3}m\perp H_2$ for holohedry. They are the following families:

(*a*) Cube oblique family XVIII, with group $m\bar{3}m\perp 2$ of order $48 \times 2 = 96$ for holohedry.

(*b*) Cube rectangle family XXII, with group $m\bar{3}m\perp mm$ of order $48 \times 2 \times 2 = 192$ for holohedry.

(*c*) Cube square family XXVI, with group $m\bar{3}m\perp 4mm$ of order $48 \times 8 = 384$ for holohedry.

(*d*) Cube hexagon family XXVII, with group $m\bar{3}m\perp 6mm$ of order $48 \times 12 = 576$ for holohedry.

These families are studied in §§4, 5, 6 and 7, respectively.

(ii) One reducible family, the (hypercube 4-dim.)-al family XXX, with group ($m\bar{3}m$.[8])$\perp m$ of order $48 \times 8 \times 2 = 768$ for holohedry.

(iii) One irreducible family, the hypercube 5-dim. family XXXII, with group $m\bar{3}m$.[8].[$\bar{5}$] of order $48 \times 8 \times 10 = 3840$ for holohedry. [$\bar{5}$] is the WPV symbol of the cyclic group generated by the product of a double rotation by angles $2\pi/5$ and $4\pi/5$ with the *homothetie* $\bar{1}_5$. The crystal cell families of space E^5 have been geometrically described by Phan (1989).

Irreducible crystal families have been studied by Weigel & Veysseyre (1991, 1993).

2.2.3. The mono cubic crystal families of space E^6. In space E^6, there exist 11 mono cubic crystal families:

(i) Five reducible families with a group of type $m\bar{3}m\perp H_3$ for holohedry. For example, the (cube hexagon)-al family, with group $m\bar{3}m\perp 6mm\perp m$ of order $48 \times 12 \times 2 = 1152$ for holohedry.

(ii) Four reducible families with a group of type ($m\bar{3}m$.[8])$\perp H_2$ for holohedry. For example, the hypercube 4-dim. oblique family, with group ($m\bar{3}m$.[8])$\perp 2$ of order $48 \times 8 \times 2 = 768$ for holohedry.

(iii) One reducible family, the (hypercube 5-dim.)-al family, with group ($m\bar{3}m$.[8].[$\bar{5}$])$\perp m$ of order $48 \times 8 \times 10 \times 2 = 7680$ for holohedry.

(iv) One irreducible family, the hypercube 6-dim. family, with group $m\bar{3}m$.[8].[$\bar{5}$].[12] of order $48 \times 8 \times 10 \times 12 = 46\,080$ for holohedry.

2.2.4. The mono cubic crystal families of space E^7. In space E^7, there exist 32 mono cubic crystal families:

(i) Twenty-one reducible families with a group of type $m\bar{3}m\perp H_4$ for holohedry, for example, the cube hexagon square family with group $m\bar{3}m\perp 6mm\perp 4mm$ of order $48 \times 12 \times 8 = 4608$ for holohedry.

(ii) Five reducible families with a group of type $(m\bar{3}m.[8])\perp H_3$ for holohedry, for example, the hypercube 4-dim. orthorhombic family with group $(m\bar{3}m.[8])\perp mmm$ of order 48 × 8 × 8 = 3072 for holohedry.

(iii) Four reducible families with a group of type $(m\bar{3}m.[8].[\bar{\bar{5}}])\perp H_2$ for holohedry, for example, the hypercube 5-dim. hexagon family with group $(m\bar{3}m.[8].[\bar{\bar{5}}])\perp 6mm$ of order 48 × 8 × 10 × 12 = 46 080 for holohedry.

(iv) One reducible family, the (hypercube 6-dim.)-al family, with group $(m\bar{3}m.[8].[\bar{\bar{5}}].[12])\perp m$ of order 48 × 8 × 10 × 12 × 2 = 92 160 for holohedry.

(v) One irreducible family, the hypercube 7-dim. family, with group $m\bar{3}m.[8].[\bar{\bar{5}}].[12].[\bar{\bar{\bar{7}}}]$ of order 48 × 8 × 10 × 12 × 14 = 645 120 for holohedry. $[\bar{\bar{\bar{7}}}]$ is the WPV symbol of the cyclic group generated by the product of a triple rotation by angles $2\pi/7$, $4\pi/7$ and $6\pi/7$ with the *homothetie* $\bar{1}_7$ of space E^7, so its order equals 14.

Remarks. The cells of the different hypercube crystal families have several cubes, but these families are in fact mono cubic crystal families because the sides of all these cubes are equal, therefore the holohedry WPV symbols cannot have more than one symbol $m\bar{3}m$. The suffix '-al' to a family name means 'hyperprism'.

3. Cubic and iso cubic point groups

In the point-group symbols of mono cubic crystal families, there appear two types of groups: the cubic point groups and the iso cubic point groups.

3.1. Cubic point groups

We recall the crystallography of space E^3.

3.1.1. Definition. A cubic point group, denoted here by g_c, is one of the five point groups of the cubic family of space E^3. Their symbols, listed in *International Tables for Crystallography* (1995) (ITC for short), are as follows:

(i) $m\bar{3}m$ and 432 of orders 48 and 24, respectively; the point group and rotation point group of the cube, respectively.

(ii) $\bar{4}3m$ of order 24 and 23 of order 12; the point group and rotation point group of the regular tetrahedron, respectively.

(iii) $m\bar{3}$ of order 24, the direct product of group 23 and the group generated by *homothetie* $\bar{1}$ (in the same way, $m\bar{3}m$ is the direct product of group 432 and *homothetie* $\bar{1}$).

3.1.2. Characteristic property of cubic point groups. These five cubic point groups contain eight or 16 operations of order 3 or 6, based on the four planes orthogonal to the four diagonals of the cube cell; these operations appear in the second term in their symbols. The set of these cubic groups can be divided into two subsets:

(i) The subset G'_c of the three groups that do not contain *homothetie* $\bar{1}$, *i.e.* groups 23, $\bar{4}3m$ and 432; g'_c denotes any group of this set.

(ii) The subset G''_c of the two centrosymmetric groups $m\bar{3}$ and $m\bar{3}m$, which contain *homothetie* $\bar{1}$; g''_c denotes any group of this set. These groups have full symbols $2/m\bar{3}$ and $4/m\,\bar{3}\,2/m$, respectively. The full symbols are very useful for defining the essential features of these groups.

In fact, these five groups belong to four mathematical structures because groups 432 and $\bar{4}3m$ are isomorphic.

Remark. The cyclic group of order 6 generated by the double rotation $2_{xy}3^{+1}_{zt}$ has the symbol $2\perp 3$; then it is impossible to confuse it with the group denoted 23, of order 12, in ITC.

3.2. Iso cubic point groups

3.2.1. Definition. An iso cubic point group, denoted $g_{\rm isoc}$ here, is any point group of space E^n with $n \geq 4$, isomorphic to one of the five cubic groups. Like the cubic groups, these groups contain eight or 16 operations of order 3 or 6 based on the four planes orthogonal to the four diagonals of the cube subcell.

3.2.2. WPV symbols of the iso cubic point groups. The WPV symbol of an iso cubic point group contains two or three subsymbols separated by a space (⊥, × or . are not used within the symbol).

(*a*) The first subsymbol is a crystallographic symmetry operation of order 2 such as 2, m, $\bar{1}$ or $\bar{1}_4$, or of order 4 such as 4, $\bar{4}$, 42, 44, $\bar{\bar{4}} = 4\times\bar{1}_5$ or $\overline{\overline{44}} = 44\times\bar{1}_5$.

(*b*) The second subsymbol is a crystallographic symmetry operation of order 3 such as 3 or 33, or of order 6 such as 6, $\bar{3}$, 62, $\bar{\bar{3}} = 3\times\bar{1}_5$, 36, 66, $\overline{\overline{33}}$ or $\overline{\overline{36}}$.

(*c*) If the symbol contains a third subsymbol, this is one of the four binary symmetry operations listed in (*a*).

3.2.3. WPV symbols of the four types of iso cubic point groups. These four types correspond to the four types of isomorphism of the cubic point groups of space E^3.

(i) *Iso cubic point groups of type 12 (of order 12)*. The iso cubic groups of type 12 are isomorphic to the cubic group 23 of mathematical type A_4, where A_n is the symbol of the alternating group of degree n and of order $n!/2$.

The WPV symbols of these groups are of the form $\alpha\ \beta$, where α is one of the three following operations of order 2: 2, m or $\bar{1}$ (their supports belong to the faces of the cubic subcell), and β is one of the ten elements of symmetry 3 or 6 listed above.

Example of an iso cubic point group of type 12. Let us consider the point group of space E^5, 2 33. This WPV symbol means that this group is an iso cubic group of order 12, for example the group 23, that it belongs to the cube hexagon family, that it has three rotations by an angle $2\pi/2$ [support planes (xy), (xz) and (yz)] and eight rotations by angles $2\pi/3$, $2\pi/3$ or $4\pi/3$, $4\pi/3$ (the rotation supports are a set of two planes; for example, a plane orthogonal to a diagonal of the cubic subcell and the plane of the hexagon cell).

(ii) *Iso cubic point groups of type 24′ (of order 24)*. The iso cubic groups of type 24′ are isomorphic to the cubic groups 432 or $\bar{4}3m$ of the mathematical type S_4, where S_n is the symbol of the symmetric group of degree n and of order $n!$.

Their symbols are of the form $\alpha\ \beta\ \gamma$, where α is one of the elements of order 4 [see (*a*)], β is one of the elements of symmetry 3 or 6 [see (*b*)] and γ is one of the four elements of binary symmetry [see (*a*)]; the supports belong to one of the following planes: $(x{+}y\ z)$, $(x{-}y\ z)$, $(z{+}x\ y)$, ..., of the cubic subcell.

Examples of iso cubic point groups of type 24′. Let us consider a point group of space E^4 of the mono cubic cubic-al family, the group that has $\overline{4}3\overline{1}$ for a WPV symbol. This group is built from the cubic group 432. Indeed, the product of the first and the last generators of group 432 with the reflection m_t transforms operation 4_{xy} into $\overline{4}_{xyt}$ and operation $2_{x+y\ z}$ into $\overline{1}_{x+y\ zt}$. This group is an iso cubic group.

Now let us consider the cubic point group $\overline{4}3m$ of space E^4 of the mono cubic cubic-al family. As previously, if we form the product of the first and the last generators of this group by the rotation 2_{tu}, we obtain from the operation $\overline{4}_{xyz}$ the operation $\overline{\overline{4}}$ and from the reflection m_{x+y} the operation $\overline{1}_{x+y\ tu}$. Then we obtain the iso cubic group $\overline{\overline{4}}3\overline{1}$, which belongs to the mono cubic cube oblique family of space E^5.

(iii) *Iso cubic point groups of type 24″ (of order 24)*. The iso cubic groups of type 24″ are isomorphic to the cubic group $m\overline{3}$ (full symbol $2/m\overline{3}$) of mathematical type $A_4 \times C_2$ (C_n is the symbol of the cyclic group of order n).

Their symbols are of the form $\alpha''\ \beta$ with $\alpha/\alpha''\ \beta$ for the full symbol. α and α'' are two elements of order 2 and β is one of the eight elements of symmetry 6 listed above. It is possible to choose three pairs of operations α and α'' because they belong to the three planes (xy), (xz) and (yz) of the cubic subcell. Each pair of these operations must satisfy the following property: the product $\alpha.\alpha''$ is the same *homothetie* $\overline{1}_n$, where n depends on the dimension space of the studied group. For example, in space E^3, if we consider group $m\overline{3}$, we have $2_{xy}.m_z = 2_{xz}.m_y = 2_{yz}.m_x = \overline{1}$, hence the property is satisfied.

Examples of iso cubic point groups of type 24″. Let us consider two point groups of the mono cubic cube hexagon family that have the same group of rotations *i.e.* group 2 33.

The WPV symbol of the first group is $m\overline{\overline{36}}$ (full symbol $2/m\overline{\overline{36}}$). The supports of the operations 2 and m are those of group $m\overline{3}$ and the double rotation reflection $\overline{\overline{36}}$ is generated by the operation $6_{x+y\ x+z}3_{tu}m_{x-y-z}$.

The WPV symbol of the second group is $2\overline{\overline{33}}$ (full symbol $\overline{1}/2\overline{\overline{33}}$) with the generators $\overline{1}_{xtu}$, 2_{yz} and $6_{x+y\ x+z}6_{tu}m_{x-y-z}$ for the double rotation reflection $\overline{\overline{33}}$, of order 6. The necessary property is verified.

Now let us consider a third example of an iso cubic group, the point group 2 62 of the mono cubic cubic-al family (full symbol 2/2 62 and alternative symbol [$23\times\overline{1}_4$]). The supports are $2_{xy}.2_{zt} = \overline{1}_4$ and $6_{x+y\ z}2_{x-y-z\ t}$ for the double rotation 62; these generators satisfy the necessary property. In space E^5, the iso cubic group 2 62 belongs to the cube rectangle family but it acts in a subspace E^4.

(iv) *Iso cubic point groups of type 48 (of order 48)*. The iso cubic groups of type 48 are isomorphic to cubic group $m\overline{3}m$ of mathematical type $S_4 \times C_2$.

Their symbols are of the form $\alpha'\ \beta\ \gamma'$ with $\alpha/\alpha'\ \beta\ \gamma/\gamma'$ for the full symbol, where α is an element of order 4, α' is an element of order 2, β is one of the elements of symmetry 3 or 6, and γ and γ' are elements of order 2, listed above. These operations must satisfy the following property: the products $\alpha^2.\alpha'$ and $\gamma.\gamma'$ must be equal to the same *homothetie* $\overline{1}_n$ for each of the three pairs of elements α, α' and for each of the six pairs of elements γ, γ'. Indeed, there exist six pairs of elements γ, γ' that belong to the planes defined by one diagonal of the faces of the cube and by the axis orthogonal to the chosen face. So in space E^3, if we consider the group $m\overline{3}m$, we have: $\alpha_2.\alpha' = 4^2_{xy}.m_z = \gamma.\gamma' = 2_{x+y\ z}.m_{x-y} = \overline{1}_{xyz}$, hence the property is satisfied.

Examples of iso cubic point groups of type 48. The following four groups are iso cubic point groups (the supports of the different point operations are given in the tables if needed):

(i) Group $m\overline{3}2$ (full symbol $\overline{4}/m\,\overline{3}\,\overline{1}/2$) of the cubic-al crystal family (space E^4) and of the cube rectangle crystal family (space E^5).

(ii) Group $m\,\overline{3}\,\overline{1}$ (full symbol $42/m\,\overline{3}\,\overline{1}_4/\overline{1}$) of the cube oblique crystal family (space E^5).

(iii) Group 2 62 m (full symbol $\overline{4}/2\,62\,\overline{1}/m$ and alternative symbol [$\overline{4}3m\times\overline{1}_4$]) of the cubic-al crystal family (space E^4).

(iv) Group $m\,\overline{\overline{36}}\,2$ (full symbol $\overline{4}/m\,\overline{\overline{36}}\,\overline{1}/2$) of the cube hexagon crystal family (space E^5).

Remark. All the iso cubic groups of the previous examples contain the reflection m. However, there are some iso cubic point groups of order 48 that do not contain this reflection, for example group 2 62 2 (full symbol 4/2 62 2/2) of the cubic-al crystal family and of the cube rectangle crystal family.

3.3. Some examples of point-symmetry groups

Here we give a list of ten point-symmetry groups with some properties. In Appendix *A*, we explain how to find the name of the family and the symbol of its holohedry, and the method for obtaining all the elements of the group from the WPV symbol.

We recall that all the elements of the cubic group $m\overline{3}m$ are described by a 3×3 matrix. However, if this group acts in a space E^5, this matrix has the form

3×3 matrix			0	0
			0	0
			0	0
0	0	0	1	0
0	0	0	0	1

All the elements of the following point groups are given in Appendix *A*.

Table 2
WPV point-group symbols of the cubic-al, cube oblique and cube triclinic families.

These three families are isomorphic. The following six iso cubic groups have an alternative symbol as written in square brackets []: $2\,62\,m$ [$\bar{4}3m\times\bar{1}_4$]; $\bar{1}\,\bar{\bar{3}}\,m$ [$\bar{4}3m\times\bar{1}_5$]; $2\,62$ [$23\times\bar{1}_4$]; $2\bar{\bar{3}}$ [$23\times\bar{1}_5$]; $2\,62\,2$ [$432\times\bar{1}_4$]; $\bar{1}\,\bar{\bar{3}}\,2$ [$432\times\bar{1}_5$]. The point-group order is easily found from the WPV symbol and knowledge of the cubic group orders. For example, the order of group $432\perp\bar{1}$ is 48 because the order of the cubic group 432 is 24 and the order of group $\bar{1}$ is 2. The order of the iso cubic group $m\bar{3}2$ is 48, the same as the order of group $m\bar{3}m$, because these groups are isomorphic. The cyclic group $\bar{\bar{3}}$ is generated by the point operation $6\times\bar{1}_5$. Examples of iso cubic point groups: (1) Point group $m\bar{3}2$ (full symbol $\bar{4}/m\,\bar{3}\,\bar{1}/2$) of the cubic-al crystal family is an iso cubic point group of order 48. The necessary property is verified because $\alpha^2.\alpha' = 4^2_{xy}.m_z = \bar{1}_{xyz} = \bar{1}_{x+y\,zt}.2_{x-y\,t}$. α and α' are defined in §3.2.3(iv). (2) Group $m\,\bar{3}\,\bar{1}$ (full symbol $42/m\,\bar{3}\,\bar{1}_4/\bar{1}$) of the cube oblique crystal family has the following supports: $4_{xy}2_{tu}$, m_z, $\bar{1}_{x+y\,ztu}$, $\bar{1}_{x-y\,tu}$. (3) Group $2\,62\,m$ (full symbol $\bar{4}/2\,62\,\bar{1}/m$) is an iso cubic group of the cube rectangle crystal family with the following supports: $\bar{4}_{xyt}$, 2_{zt}, $\bar{1}_{x+y\,zt}$, m_{x-y}.

Cubic-al family (E^4)		Cube oblique family (E^5)		Cube triclinic family (E^6)	
System 19_1 (5 point groups)		Subfamily XVIIIa (5 point groups)		Subfamily 21_1 (5 point groups)	
$2\,62\,m$ (hol.)		$\bar{1}\,\bar{\bar{3}}m$ (hol.)		$\bar{4}3m\times\bar{1}_6$ (hol.)	
23	$2\,62$	23	$2\bar{\bar{3}}$	23	$23\times\bar{1}_6$
$\bar{4}3m$	$\bar{4}3\bar{1}$	$\bar{4}3m$	$42\,3\,\bar{1}_4$	$\bar{4}3m$	$\bar{\bar{4}}3\bar{1}_5$
System 19_2 (11 point groups)		Subfamily XVIII (11 point groups)		Subfamily 21_2 (11 point groups)	
$m\bar{3}m\perp m$ (hol.)		$m\bar{3}m\perp 2$ (hol.)		$m\bar{3}m\perp\bar{1}$ (hol.)	
$23\perp m$	$\bar{4}3m\perp m$	$23\perp 2$	$\bar{4}3m\perp 2$	$23\perp\bar{1}$	$\bar{4}3m\perp\bar{1}$
$m\bar{3}$	$m\bar{3}\perp m$	$m\bar{3}$	$m\bar{3}\perp 2$	$m\bar{3}$	$m\bar{3}\perp\bar{1}$
432	$432\perp m$	432	$432\perp 2$	432	$432\perp\bar{1}$
$2\,62\,2$	$42\,3\,2$	$\bar{1}\,\bar{\bar{3}}\,2$	$\bar{\bar{4}}3\bar{1}$	$432\times\bar{1}_6$	$422\,3\,\bar{1}_4$
$m\bar{3}m$	$m\bar{3}2$	$m\bar{3}m$	$m\,\bar{3}\,\bar{1}$	$m\bar{3}m$	$m\,\bar{3}\,\bar{1}_4$

(*a*) Group $\bar{4}3m\perp 2$.

(*b*) Group $2\bar{\bar{3}}$ or $23\times\bar{1}_5$ as alternative symbol. $\bar{\bar{3}}$ is the symbol of a cyclic group of order 6, generated by the inversion rotation $3\times\bar{1}_5$, 3 being the rotation by an angle $2\pi/3$ in the plane orthogonal to one diagonal of the cube [space (*xyz*)].

(*c*) Group $432\perp mm$ (short notation for $432\perp m\perp m$). The subgroup 432 acts in space (*xyz*), the support of the first *m* is space *t* and the support of the second *m* is space *u*; moreover the two axes *t* and *u* are orthogonal and orthogonal to space (*xyz*).

(*d*) Group $m\bar{3}\perp m$. This group acts in space E^5. The reflection *m* belongs to axis *t* or *u*.

(*e*) Group (42 3 2).2. Group 42 3 2 is an iso cubic group isomorphic to group $\bar{4}3m$ (order 24) which belongs to the cube-al family (space E^4) and to the cube rectangle family (space E^5). Indeed, the product of the operation $\bar{4}$ of group $\bar{4}3m$ with the reflection m_t gives the double rotation 42 in space (*xyzt*) and the product of the two reflections m_{x+y} and m_t gives the rotation $2_{x+y\ t}$, hence group 42 3 2. The 24 elements of group 42 3 2 are as follows: identity; nine elements 2 (2_{xy}, 2_{xz}, 2_{yz}, $2_{x\pm y\ t}$, $2_{x\pm z\ t}$, $2_{y\pm z\ t}$); eight elements 3 ($3^{\pm 1}_{x\pm y\ x\pm z}$); and six elements 42 ($4^{\pm 1}_{xy}2_{zt}$, $4^{\pm 1}_{xz}2_{yt}$ and $4^{\pm 1}_{yz}2_{xt}$). The second factor 2, after the dot ., acts in the plane (*tu*).

(*f*) Group $m\bar{3}\perp 4mm$. Group $m\bar{3}$ is a cubic group acting in space (*xyz*) whereas 4*mm* is the symmetry group of the square acting in plane (*tu*).

(*g*) Group (2 62).44. Group 2 62 is an iso cubic group (full symbol 2/2 62) defined in §3.2.3(iii) isomorphic to point group $m\bar{3}$ (full symbol $2/m\bar{3}$). It belongs to the cube-al family (space E^4) and to the cube rectangle family (space E^5).

(*h*) Group $\bar{1}_4\,36\,\bar{1}$ (full symbol $\bar{4}/\bar{1}_4\,36\,\bar{1}/\bar{1}$).

(*i*) Group 23.46. Group 46 is a cyclic group of order 12.

(*j*) Group (42 3 2).3*m*.

4. WPV symbols of the 16 point groups of the three isomorphic mono cubic crystal families cubic-al, cube oblique and cube triclinic

Each of the mono cubic crystal families cubic-al (No. 19, space E^4), cube oblique (No. XVIII, space E^5) and cube triclinic (No. 21, space E^6) have 16 point groups, listed in Table 2. As we can see in this table, these point groups are isomorphic. Holohedry is abbreviated to (hol.) in all the tables.

The holohedries of the primitive crystal families are of the form $m\bar{3}m\perp\bar{1}_p$ where $p = n - 3$, n being the dimensional space with the special cases $\bar{1}_1 = m$, $\bar{1}_2 = 2$ and $\bar{1}_3 = \bar{1}$; the order of all these holohedries is $96 = 48\times 2$. For each family, we find:

(i) The five cubic groups g_c.

(ii) The five groups $g_c\perp m$ for family 19, the five groups $g_c\perp 2$ for family XVIII and the five groups $g_c\perp\bar{1}$ for family XXI.

(iii) Six iso cubic groups as explained below.

The six iso cubic groups of the cubic-al family (No. 19) are as follows:

(i) Three iso cubic groups with $g'_c\times\bar{1}_4$ for alternative symbol (g'_c is defined in §3.1.2). They are the following groups: 2 62 or [$23\times\bar{1}_4$], 2 62 2 or [$432\times\bar{1}_4$] and $2\,62\,m$ or [$\bar{4}3m\times\bar{1}_4$] [see §§3.2.3(iii) and (iv)].

(ii) Three iso cubic groups: $\bar{4}3\bar{1}$, 42 3 2 (isomorphic to group $\bar{4}3m$) and $m\bar{3}2$ (isomorphic to group $m\bar{3}m$).

Let G_4 denote the set of these six iso cubic groups of the cubic-al family (space E^4).

Table 3
WPV point-group symbols of the cube rectangle family (No. XXII, space E^5).

The following groups have an alternative symbol as written in square brackets []: (2 62)$\perp m$ [$(23\times\bar{1}_4)\perp m$]; (2 62).2 [$(23\times\bar{1}_4)$.2]; (2 62 m).2 [$(\bar{4}3m\times\bar{1}_4)$.2]; (2 62 m)$\perp m$ [$(\bar{4}3m\times\bar{1}_4)\perp m$]; (2 62 2)$\perp m$ [$(432\times\bar{1}_4)\perp m$]; (2 62 2).2 [$(432\times\bar{1}_4)$.2]; $\bar{1}\bar{3}2$ [$(42\,3\,2)\times\bar{1}_5$]; $2\,62\,\bar{1}_4$ [(42 3 2)×2]. The other groups are listed in Table 2. Examples of iso cubic groups: (1) Group $2\,62\,\bar{1}_4$ (full symbol $42/2\,62\,\bar{1}_4/2$) is an iso cubic group isomorphic to group $m\bar{3}m$, obtained through the following process: $m_x.m_t = 2_{xt}$, $\bar{3}.m_u = 62$, $2_{x+y\ z}.2_{tu} = \bar{1}_{x+y\,ztu}(\bar{1}_4)$, hence its symbol. (2) Group $2\,62\,\bar{1}$ (full symbol $\bar{4}/2\,62\,\bar{1}/\bar{1}$) is an iso cubic group isomorphic to group $m\bar{3}m$. The geometric support of rotation 2 after the symbol . is the plane (tu) and the geometric support of the double rotation 42 is the space ($xyzt$).

System 19_1 (5 point groups)	Subfamily XXIIa (11 point groups)				
E^4	$E^3 + E^1$	$E^3 + E^1 + E^1$	E^4	$E^4 + E^1$	E^5
23	$23\perp m$		2 62		
2 62				(2 62)$\perp m$	
$\bar{4}3m$	$\bar{4}3m\perp m$				
2 62 m (hol.)			2 62 m	(2 62 m)$\perp m$ (hol.)	
$\bar{4}3\bar{1}$			$\bar{4}3\bar{1}$	$(\bar{4}3\bar{1})\perp m$	$\bar{1}\bar{3}2$
			42 3 2		$2\,62\,\bar{1}_4$

System 19_2 (11 point groups)	Subfamily XXII (20 point groups)				
E^4	$E^3 + E^1$	$E^3 + E^1 + E^1$	E^4	$E^4 + E^1$	E^5
$23\perp m$		$23\perp mm$			(2 62).2
$\bar{4}3m\perp m$		$\bar{4}3m\perp mm$			(2 62 m).2
					$(\bar{4}3\bar{1})$.2
					$2\,62\,\bar{1}$
432					
$432\perp m$	$432\perp m$	$432\perp mm$			
2 62 2			2 62 2	(2 62 2)$\perp m$	(2 62 2).2
42 3 2				(42 3 2)$\perp m$	(42 3 2).2
$m\bar{3}$					
$m\bar{3}\perp m$					
$m\bar{3}m$	$m\bar{3}\perp m$	$m\bar{3}\perp mm$			
$m\bar{3}m\perp m$	$m\bar{3}m\perp m$	$m\bar{3}m\perp mm$ (hol.)			
$m\bar{3}2$			$m\bar{3}2$	$(m\bar{3}2)\perp m$	$(m\bar{3}2)$.2

The six iso cubic groups of the cubic oblique family (No. XVIII) are as follows:

(i) Three iso cubic groups with $g'_c\times\bar{1}_5$ for the alternative symbol (g'_c is defined in §3.1.2). They are the following groups: $2\bar{\bar{3}}$ ([$23\times\bar{1}_5$], full symbol $\bar{1}/2\bar{\bar{3}}$), $\bar{1}\bar{\bar{3}}2$ ([$432\times\bar{1}_5$], full symbol $4/\bar{1}\bar{\bar{3}}\bar{1}/2$) and $\bar{1}\bar{\bar{3}}m$ ([$\bar{4}3m\times\bar{1}_5$], full symbol $\bar{4}/\bar{1}\bar{\bar{3}}\bar{1}/m$).

(ii) Three iso cubic groups obtained from the rotation 2_{tu}: $42\,3\,\bar{1}_4$ obtained from group 432, $\bar{\bar{4}}3\bar{1}$ obtained from group $\bar{4}3m$ and $m\,\bar{3}\,\bar{1}$ obtained from group $m\bar{3}m$.

Among these six groups, two belong to type 24′, $42\,3\,\bar{1}_4$ and $\bar{\bar{4}}3\bar{1}$, one belongs to type 24″, $2\bar{\bar{3}}$, and three belong to type 48, $\bar{1}\bar{\bar{3}}2$, $\bar{1}\bar{\bar{3}}m$ and $m\,\bar{3}\,\bar{1}$.

Let G_5 denote the set of these six iso cubic point groups of the cube oblique crystal family, isomorphic to the six iso cubic point groups of the set G_4 of the cubic-al family.

The six iso cubic point groups of the cube triclinic family (space E^6) are studied in a similar way. For instance, the occurrence of the *homothetie* $\bar{1}$ which acts in space (tuv) gives the following three iso cubic groups: $\bar{\bar{4}}3\bar{1}_5$, $422\,3\,\bar{1}_4$ and $m\,\bar{3}\,\bar{1}_4$. Then there are six groups of the same type, $\bar{4}3m\times\bar{1}_6$, $23\times\bar{1}_6$ and $432\times\bar{1}_6$.

5. The mono cubic cube rectangle crystal family of space E^5

The mono cubic cube rectangle crystal family (No. XXII) of space E^5 has 31 point groups determined by the computer program (Veysseyre & Veysseyre, 2002). These 31 point groups belong to two subfamilies: subfamily XXIIa with group (2 62 m)$\perp m$ of order 96 for holohedry and subfamily XXII with group $m\bar{3}m\perp mm$ of order 192 for holohedry. The cell of this family is built from a cube (space E^3) and a rectangle (space E^2), all the axes x, y, z, t and u are orthogonal two by two.

To be able to predict all the point groups of this family geometrically, we must return to the cubic-al family of space E^4.

5.1. The cubic-al family of space E^4

This family splits into two subfamilies and their point groups are listed in the first columns of Tables 2 and 3. Here we summarize the results already given and we explain how all these groups have been obtained.

5.1.1. The tetrahedron-al subfamily (system 19_1). This subfamily has five point groups:

(i) Two cubic groups 23 (the rotation group of the regular tetrahedron) and $\bar{4}3m$ (the group of the regular tetrahedron). The products of these two groups with *homothetie* $\bar{1}_4$ give two iso cubic groups $23\times\bar{1}_4$ or 2 62 and $\bar{4}3m\times\bar{1}_4$ or 2 62 m. The first symbol explains the construction of the group very well and the second one is the symbol of an iso cubic group.

(ii) One iso cubic group $\bar{4}3\bar{1}$ isomorphic to group $\bar{4}3m$.

5.1.2. The cubic-al subfamily (system 19_2). This subfamily has 11 point groups:

(i) The three other cubic groups 432, $m\bar{3}$ and $m\bar{3}m$, and their orthogonal products with operation m of the space defined by axis u: $432\perp m$, $m\bar{3}\perp m$, $m\bar{3}m\perp m$ and the direct product $432\times\bar{1}_4$ or 2 62 2; hence seven groups.

(ii) The two orthogonal products $23\perp m$ and $\bar{4}3m\perp m$; operation m is defined as previously. Let us note an error in Janssen *et al.* (1999): in Table 3, the system given as $m\bar{4}3m\perp m$ should read $m\bar{3}\perp m$.

(iii) Two iso cubic groups: 42 3 2 isomorphic to 432 and $m\bar{3}2$ isomorphic to $m\bar{3}m$.

Groups of systems 19_1 and 19_2 are listed in the first column of Table 3.

5.2. The cube rectangle family (No. XXII) of space E^5

This family splits into two subfamilies:

(i) Subfamily XXIIa, which has 11 point groups. Group $(2\ 62\ m)\perp m$ of order 96 = 48 × 2 is the holohedry.

(ii) Subfamily XXII, which has 20 point groups. Group $m\bar{3}m\perp mm$ of order 192 = 48 × 2 × 2 is the holohedry.

5.2.1. From system 19_1 of space E^4 to subfamily XXIIa of space E^5. Of the two mirrors of the rectangle (m_t and m_u), only one, m_u, can appear as generator in the second factor of the point-group symbols of subfamily XXIIa. The first part of Table 3 shows how, owing to the geometry, it is possible to predict the 11 point groups of this family. We find:

(i) Six groups built from the following two groups of the regular tetrahedron: 23 and $\bar{4}3m$. They are groups $23\perp m$ and $\bar{4}3m\perp m$, 2 62 (or $23\times\bar{1}_4$), 2 62 m (or $\bar{4}3m\times\bar{1}_4$), and lastly $(2\ 62)\perp m$ and $(2\ 62\ m)\perp m$.

(ii) Three groups built from group $\bar{4}3\bar{1}$. They are groups $\bar{4}3\bar{1}$, $(\bar{4}3\bar{1})\perp m$ and $(\bar{4}3\bar{1})\times\bar{1}_5$ or $\bar{1}\,\bar{3}\,2$ as iso cubic group symbol.

(iii) Two iso cubic groups: 42 3 2 and $(42\,3\,2)\times\bar{1}_4$ or $2\ 62\ \bar{1}_4$.

5.2.2. From system 19_2 of space E^4 to subfamily XXII of space E^5. The two mirrors of the rectangle (m_t and m_u) as well as the rotation 2_{tu} can appear as generators for some point groups. Hence we find (second part of Table 3):

(i) Five point groups $g_c\perp mm$, where g_c is one of the five cubic groups (third column).

(ii) Three point groups $g_c\perp m$; two groups of this type belong to family XXIIa (second column).

(iii) Six groups built from the six iso cubic point groups of family cubic-al (set G_4 defined in §4). These six groups are the product of each group of set G_4 with a rotation of order 2 in the plane (tu) (sixth column).

(iv) Two iso cubic point groups built from the cubic group 432: 2 62 2 and $(2\ 62\ 2)\perp m$, respectively $(432\times\bar{1}_4)$ and $(432\times\bar{1}_4)\perp m$ (fourth and fifth columns).

(v) Two point groups $m\bar{3}2$ and $(m\bar{3}2)\perp m$ (fourth and fifth columns).

(vi) One group $(42\ 3\ 2)\perp m$ (fifth column).

(vii) One iso cubic group $2\ 62\ \bar{1}$ isomorphic to group $m\bar{3}m$ (sixth column).

6. The mono cubic cube square crystal family of space E^5

The holohedry symbol of the mono cubic cube square crystal family (No. XXXVI) of space E^5 is group $m\bar{3}m\perp 4mm$ of order 48 × 8 = 384. As previously, we used the computer program and a geometrical method to obtain the 31 point groups of this family and their WPV symbols.

We start by recalling the following.

The square crystal family of space E^2 has only two groups with rotations of order 4 *i.e.* groups 4 and 4mm (holohedry).

The cubic crystal family of space E^3 has five cubic groups g_c; this set of cubic groups can be divided into two subsets (see §3.1.2). Two groups of this family, 23 and $m\bar{3}$, do not contain a rotation of order 4.

The cubic-al crystal family of space E^4 has 16 groups: the five cubic groups g_c, the five groups $g_c\perp m$ and the set G_4 of the six iso cubic groups of the cubic-al family; among these, only one group, 2 62, does not contain a rotation of order 4.

6.1. Point groups of the mono cubic cube square crystal family

Any point-group symbol of the mono cubic cube square crystal family (space E^5) must contain:

(i) A first part that is the symbol of a group of space E^3 or E^4. This symbol is either a cubic group g_c, or an iso cubic group of the set G_4, or a group of type $g_c\perp m$.

(ii) A second part that is the symbol of a group of order 4 (4, $\bar{\bar{4}}$, 44, $\overline{\overline{44}}$) or of order 8 (only 4mm).

These properties define any group of this family perfectly.

Remark. The second symbol 44 or $\overline{\overline{44}}$ cannot appear with a first symbol that already contains one symmetry of order 4. In fact, this point group would be a group of a mono cubic crystal family of space E^7 or E^8. For example, a point group $g\perp 44$ (g being an element of the set G_4) such as group $(42\ 3\ 2)\perp 44$ belongs to the (cubic-al) diclinic di squares crystal family (space E^8). However, if the first part of the symbol does not contain any rotation of order 4, then the second one can be a double rotation of order 4 such as 44 or an inverse double rotation of order 4 such as $\overline{\overline{44}}$.

The 31 point groups of the cube square crystal family are listed in Table 4. We find:

(i) Five groups of the form $g_c\perp 4$ and five groups $g_c\perp 4mm$, where g_c is any cubic group (first column).

(ii) Six groups of the form $g.4$, where g is any group of the set G_4 (second column).

(iii) Three groups $g'_c.\bar{\bar{4}}$, three groups $(g'_c\perp m).\bar{\bar{4}}$ and one unique group $(\bar{4}3\bar{1}).\bar{\bar{4}}$ that has not been counted before, where g'_c is an element of the set G'_c (third column). The set G'_c is defined in §3.1.2.

Table 4
WPV point-group symbols of the cube square family (No. XXVI, space E^5).

The following iso cubic groups have an alternative symbol as written in square brackets []: (2 62).4 [$(23\times\bar{1}_4)$.4]; (2 62).44 [$(2\,3\times\bar{1}_4)$.44]; (2 62).$\overline{\overline{44}}$ [$(2\,3\times\bar{1}_4).\overline{\overline{44}}$]; (2 62 m).4 [$(\bar{4}3m\times\bar{1}_4)$.4]; $(\bar{4}3\bar{1}).\bar{\bar{4}}$ [$(42\,3\,2).\bar{\bar{4}}$]; (2 62 2).4 [$\{(4\,3\,2)\times\bar{1}_4\}$.4].

Family XXVI
(31 point groups)

$E^3 + E^2$	$E^5(E^4.E^2)$	E^5	E^5	E^5
23⊥4	(2 62).4	$23.\bar{\bar{4}}$	23.44	$23.\overline{\overline{44}}$
23⊥4mm		$(23\perp m).\bar{\bar{4}}$	$(23\perp m)$.44	$(23\perp m).\overline{\overline{44}}$
			(2 62).44	$(2\,62).\overline{\overline{44}}$
$\bar{4}3m\perp 4$	(2 62 m).4	$\bar{4}3m.\bar{\bar{4}}$		
$\bar{4}3m\perp 4mm$		$(\bar{4}3m\perp m).\bar{\bar{4}}$		
	$(\bar{4}3\bar{1})$.4	$(\bar{4}3\bar{1}).\bar{\bar{4}}$		
432⊥4	(2 62 2).4	$432.\bar{\bar{4}}$		
432⊥4mm	(42 3 2).4	$(432\perp m).\bar{\bar{4}}$		
$m\bar{3}\perp 4$			$m\bar{3}$.44	
$m\bar{3}\perp 4mm$			$(m\bar{3}\perp m)$.44	
$m\bar{3}m\perp 4$	$(m\bar{3}2)$.4			
$m\bar{3}m\perp 4mm$ (hol.)				

The following groups, in columns four and five:

(iv) Two groups g''_c.44, 23.44 and $m\bar{3}$.44, and one unique group $23.\overline{\overline{44}}$, because $m\bar{3}.44 = m\bar{3}.\overline{\overline{44}}$ due to the occurrence of the *homothetie* $\bar{1}$, g''_c being an element of the set G''_c. The set G''_c is defined in §3.1.2.

(v) Two groups $(g''_c\perp m)$.44, $(23\perp m)$.44 and $(m\bar{3}\perp m)$.44, and one unique group, $(23\perp m).\overline{\overline{44}}$.

(vi) Two groups (2 62).44 and $(2\,62).\overline{\overline{44}}$.

Theorem. All the groups of the cube square crystal family act in space E^5. Indeed, all the point groups of the first part of the symbol act in space E^3 at least, those of the second part of the symbol act in space E^2 at least and these two spaces are orthogonal and disjoint.

Remark. The symbol $g_c\perp m$ in space E^5 implies that there are two mirrors m_t and m_u and possibly other mirrors belonging to group g_c.

Table 5
WPV point-group symbols of the cube hexagon family (No. XXVII, space E^5).

The following iso cubic groups have an alternative symbol as written in square brackets []: $2\overline{\overline{33}}$ [$(2\,33)\times\bar{1}_5$]; $(\bar{1}\,\overline{\overline{33}}\,2)$ [$(42\,33\,2)\times\bar{1}_5$]; $23.\bar{3}$ [$(23\perp 3)\times\bar{1}_5$]; $\bar{4}3m.\bar{3}$ [$(\bar{4}3m\perp 3)\times\bar{1}_5$]; $432.\bar{3}$ [$(432\perp 3)\times\bar{1}_5$]; $23.\bar{3}m$ [$(23\perp 3m)\times\bar{1}_5$]; $\bar{4}3m.\bar{3}m$ [$(\bar{4}3m\perp 3m)\times\bar{1}_5$]; $432.\bar{3}m$ [$(432\perp 3m)\times\bar{1}_5$]; (2 62 m).3 [$(\bar{4}3m\times\bar{1}_4)$.3]. For $m\overline{\overline{36}}$ (full symbol $2/m\overline{\overline{36}}$) $\overline{\overline{36}}$ is generated by the double rotation reflection $6^1_{x+y\,x+z}3^1_{tu}m_{x-y-z}$. $m\overline{\overline{36}}2$ (full symbol $\bar{4}/m\,\overline{\overline{36}}\,\bar{1}/2$) is an iso cubic group with $\bar{4}_{xyt}$, m_z, $\bar{1}_{x+y\,zt}$, m_x and $6^1_{x+y\,x+z}3^1_{tu}m_{x-y-z}$ for generators.

Subfamily XXVIIa
(12 point groups)

E^5	E^5	E^5
2 33	2 36	$2\overline{\overline{33}}$
$\bar{4}\,33\,\bar{1}$	$\bar{1}_4\,36\,\bar{1}$	
42 33 2	$\bar{1}_4\,36\,2$	$\bar{1}\,\overline{\overline{33}}\,2$
$m\overline{\overline{36}}$		$(m\overline{\overline{36}})\times\bar{1}_5$
$m\,\overline{\overline{36}}\,2$		$(m\,\overline{\overline{36}}\,2)\times\bar{1}_5$ (hol.)

Subfamily XXVII
(47 point groups)

$E^3 + E^2$	$E^3 + E^2$	E^5	E^5
23⊥3	23⊥6	$23.\bar{3}$	
23⊥3m	23⊥6mm	$23.\bar{3}m$	
		(2 62).3	(2 62).6
		$23.\overline{\overline{43}}$	23.46
$\bar{4}3m\perp 3$	$\bar{4}3m\perp 6$	$\bar{4}3m.\bar{3}$	
$\bar{4}3m\perp 3m$	$\bar{4}3m\perp 6mm$	$\bar{4}3m.\bar{3}m$	
		(2 62 m).3	(2 62 m).6
		$(\bar{4}3\bar{1})$.3	$(\bar{4}3\bar{1})$.6
		$(\bar{4}3\bar{1})$.3m	
		$(\bar{\bar{4}}3\bar{1})$.3	
432⊥3	432⊥6	$432.\bar{3}$	
432⊥3m	432⊥6mm	$432.\bar{3}m$	
		(2 62 2).3	(2 62 2).6
		(42 3 2).3	(42 3 2).6
		(42 3 2).3m	
		$(\bar{1}\,\bar{3}\,2)$.3	
		$(42\,3\,\bar{1}_4)$.3	
$m\bar{3}\perp 3$	$m\bar{3}\perp 6$		
$m\bar{3}\perp 3m$	$m\bar{3}\perp 6mm$		
$m\bar{3}m\perp 3$	$m\bar{3}m\perp 6$	$(m\bar{3}2)$.3	$(m\bar{3}2)$.6
$m\bar{3}m\perp 3m$	$m\bar{3}m\perp 6mm$ (hol.)	$(m\bar{3}2)$.3m	$(m\,3\,\bar{1})$.3

7. The mono cubic cube hexagon crystal family of space E^5

The 59 point groups of the cube hexagon crystal family (No. XXVII) are listed in Table 5. The cell of this family is built from a cube (space E^3) and one hexagon (space E^2, orthogonal to space E^3). As previously, we used the geometrical method to give WPV symbols to these 59 point groups as defined by the computer program. Family XXVII splits into two subfamilies: subfamily XXVIIa with 12 point groups [holohedry is group $(m\,\overline{\overline{36}}\,2)\times\bar{1}_5$ of order 96] (generators of group $m\,\overline{\overline{36}}\,2$ are given in Table 5) and subfamily XXVII with 47 point groups (holohedry is group $m\bar{3}m\perp 6mm$ of order 48 × 12 = 576).

As pointed out by the family name, each point group contains a direct or inverse double rotation by angles $2\pi/3$, $4\pi/3$ or $2\pi/6$, $5\pi/6$, for example. Moreover, they contain three elements of order 2 or 4, the supports of these belonging to the three faces of the cubic subcell.

This double rotation is:

(i) either simultaneous, in the pair of planes p_1 and p_2 for the 12 point groups of subfamily XXVIIa, ten of these being iso cubic groups except groups $(m\,\overline{\overline{36}})\times\bar{1}_5$ and $(m\,\overline{\overline{36}}\,2)\times\bar{1}_5$;

(ii) or built from two independent simple rotations in the planes p_1 and p_2. The 47 point groups of the primitive subfamily XXVII obtained in this way do not contain iso cubic groups [because the simple rotation in plane (tu) is incompatible with the occurrence of iso cubic groups].

7.1. Point groups of the subfamily XXVIIa

Point groups of this family are listed in the first part of Table 5. The double rotations that generate the point groups are the following four: 33, 36, $\overline{\overline{33}} = 33\times\overline{1}_5 = 66m$ and $\overline{\overline{36}} = 36\times\overline{1}_5 = 63m$. Among the cyclic groups generated by these double rotations, only one group contains the *homothetie* $\overline{1}_5$: the group $\overline{\overline{33}}$.

These four double rotations generate ten iso cubic groups and two no iso cubic groups:

(i) One iso cubic group of type 12: 2 33 (first column).

(ii) Two iso cubic groups of type 24′: $\overline{4}\,33\,\overline{1}$ (isomorphic to group $\overline{4}3\overline{1}$) and 42 33 2 (isomorphic to group 42 3 2) (first column).

(iii) Three iso cubic groups of type 24″: 2 36 (full symbol $\overline{1}_4/2\,36$) (first column), $2\,\overline{\overline{33}}$ (full symbol $\overline{1}/2\,\overline{\overline{33}}$) (third column) and $m\,\overline{\overline{36}}$ (full symbol $2/m\,\overline{\overline{36}}$) (first column).

(iv) Four iso cubic groups of type 48: $\overline{1}_4\,36\,2$ (full symbol $42/\overline{1}_4\,36\,2/2$), $\overline{1}_4\,36\,\overline{1}$ (full symbol $\overline{4}/\overline{1}_4\,36\,\overline{1}/\overline{1}$) (second column), $m\,\overline{\overline{36}}\,2$ (full symbol $\overline{4}/m\,\overline{\overline{36}}\,\overline{1}/2$) (first column) and $\overline{1}\,\overline{\overline{33}}\,2$ (full symbol $\overline{4}/\overline{1}\,\overline{\overline{33}}\,\overline{1}/2$ and alternative symbol $[(42\,33\,2)\times\overline{1}_5]$) (third column).

(v) Two no iso cubic groups: $(m\,\overline{\overline{36}})\times\overline{1}_5$ and $(m\,\overline{\overline{36}}\,2)\times\overline{1}_5$ (third column).

In conclusion, the WPV symbol of each of the 12 point groups of subfamily XXVIIa has one unique iso cubic factor group whose eight or 16 operations of order 3 or 6 are the double rotations 33, 36, $\overline{\overline{36}}$ or $\overline{\overline{33}}$. These double-rotation supports are one of the planes orthogonal to one diagonal of the cubic subcell and the plane of the hexagon. Two groups are the product of an iso cubic group with the *homothetie* $\overline{1}_5$.

Any symbol verifying this property defines a point group belonging to subfamily XXVIIa. Therefore, all the groups containing the cyclic group generated by the double rotation $\overline{\overline{36}}$ obviously act in space E^5.

7.2. Point groups of the subfamily XXVII

The WPV symbols of the 47 point groups of family XXVII are made up of the product of a first cubic or iso cubic group, acting in spaces (*xyz*), (*xyzt*) or (*xyztu*), and a second factor. This second factor is one of the four groups 3, 3*m*, 6 or 6*mm*, acting in plane (*tu*), or one of the two groups $\overline{3}$ or $m\overline{3}$, acting in space (*tux*). The two groups 23.46 and $23.\overline{\overline{43}}$ that are an exception to this rule are the product of group 23, one of the two groups g_c that do not contain a rotation of order 4, and of one cyclic group 46 and $\overline{\overline{43}}$. The converse of this rule is true.

To simplify, we use the following notations:

G'_4 is the subset of the three point groups of the set G_4 (see §4) that do not contain the *homothetie* $\overline{1}_4$, *i.e.* groups 42 3 2, $\overline{4}3\overline{1}$ and $m\overline{3}2$.

G'_5 is the subset of the three point groups of the set G_5 (see §4) that do not contain the *homothetie* $\overline{1}_5$, *i.e.* groups $42\,3\,\overline{1}_4$, $\overline{4}3\overline{1}$ and $m\,\overline{3}\,\overline{1}$.

Now it is possible to classify geometrically almost all these 47 point groups (Table 5, second part). In fact, we must obtain:

(i) Five groups $g_c\perp 3$, five groups $g_c\perp 3m$, five groups $g_c\perp 6$ and five groups $g_c\perp 6mm$, therefore 20 point groups (first and second columns).

(ii) Three groups $g'_c.\overline{3}$ ($[(g'_c\perp 3)\times\overline{1}_5]$ as alternative symbol) and three groups $g'_c.\overline{3}m$ ($[(g'_c\perp 3m)]\times\overline{1}_5$ as alternative symbol), therefore six point groups (third column). The three groups g'_c are the three cubic groups that are not centrosymmetric.

(iii) Six groups *g*.3 (third column), six groups *g*.6 (fourth column), where *g* is an element of the set G_4, and three groups $g'.3m$ (third column), where g' is an element of the set G'_4, therefore 15 point groups (the six groups $g'.6mm$ have already been obtained).

(iv) Three groups $g'_5.3$, where g'_5 is an element of the set G'_5, and one unique group $(\overline{1}\,\overline{3}\,2).3$ (fourth column). The other groups, the product of a group of the set G'_5 with groups 3*m*, 6 or 6*mm*, have been listed with a different symbol.

(v) The two groups $23.\overline{\overline{43}}$ and 23.46.

Remark. The cube square crystal family and cube hexagon crystal subfamily, No. XXVII of space E^5, do not have iso cubic point groups. The orders of most of the point groups of these families are higher than 48.

8. The last two mono cubic crystal families of space E^5: the (hypercubic 4-dim.)-al and the hypercubic 5-dim. crystal families

We recall that the mono cubic hypercubic crystal family (No. XXIII) of space E^4 splits into two subfamilies, which is an exceptional case of space E^4 (Veysseyre *et al.*, 1984; Weigel *et al.*, 1987):

(i) The body-centred family XXIIIa. Its holohedry is group $m\overline{3}m.[12].2$ of order 48 × 12 × 2 = 1152. We denote by $G'_{hc/4}$ the set of the 16 point groups of this family.

(ii) The primitive subfamily XXIII. Its holohedry is group $m\overline{3}m.[8]$ of order 48 × 8 = 384. We denote by $G_{hc/4}$ the set of the 21 point groups of this family.

8.1. The (hyper-cubic 4-dim.)-al crystal family of space E^5

This family is called family 23 in Janssen *et al.* (1999) and family XXVIII in Veysseyre & Veysseyre (2002) and Plesken (1981). It splits into two subfamilies:

(i) The primitive subfamily XXVIII. This has group $(m\overline{3}m.[8])\perp m$ of order 48 × 8 × 2 = 768 for holohedry and 90 point groups. Among these groups, we find the set $G_{hc/4}$ of 21 groups acting in space E^4, then 21 groups $g_{hc/4}\perp m$ ($g_{hc/4}$ being an element of set $G_{hc/4}$) acting in space $E^4 + E^1$, and 48 groups acting in space E^5.

(ii) The centred subfamily XXVIIIa. This has group $(m\overline{3}m.[12].2)\perp m$ of order 48 × 12 × 2 × 2 = 2304 for holohedry and 51 point groups. Among these groups, we find the set $G'_{hc/4}$ of 16 groups, then the 16 groups $g'_{hc/4}\perp m$ ($g'_{hc/4}$ being an element of set $G'_{hc/4}$) acting in space $E^4 + E^1$, and 19 groups acting in space E^5.

8.2. The hypercubic 5-dim. crystal family of space E^5

The hypercubic 5-dim. family has group $m\bar{3}m.[8].[\bar{5}]$ of order 48 × 8 × 10 = 3840 for holohedry and 13 point groups whose elements were defined using the computer program.

9. Di cubic crystal families of spaces E^6, E^7 and E^8

9.1. Definition

A crystal family of space E^n is called di cubic if its cell is built from two cubic cells with unequal side lengths, or a cube and a hypercube with unequal side lengths, or two hypercubes with unequal side lengths belonging to two disjoint subspaces.

9.2. Characteristic property

The WPV symbol of the holohedry of a di cubic crystal family contains two and only two symbols $m\bar{3}m$, possibly with some other factors such as 2, *m etc*.

9.3. Examples

The di cubic crystal families appear only in spaces of dimension n equal to or greater than 6.

In space E^6, there exist two di cubic crystal families:

(i) The monoclinic di cubic crystal family. Its holohedry is group $m\bar{3}m(m\bar{3}m)$ or 2 662 2 of order 48. Its cell is defined by three parameters: two length parameters (the sides of the two cubes) and one angle parameter. This family is No. 62 in Table 8 of Janssen *et al.* (1999) and No. LXVI in Plesken & Hanrath (1984).

(ii) The di cubes crystal family. Its holohedry is group $m\bar{3}m\perp m\bar{3}m$ of order 48^2 = 2304. Its cell is defined by two geometrical parameters: the side lengths of the two cubes, belonging to two orthogonal subspaces. This family is No. 80 in Table 8 of Janssen *et al.* (1999) and No. LXVI in Plesken & Hanrath (1984).

In space E^7, there exist three di cubic crystal families:

(i) The (monoclinic di cubic)-al crystal family. Its holohedry is group $m\bar{3}m(m\bar{3}m)\perp m$ of order 48 × 2 = 96. Its cell is defined by four geometrical parameters: three lengths (the sides of the two cubes and the height of the hyperprism) and one angle.

(ii) The (di cubes)-al crystal family. Its holohedry is group $m\bar{3}m\perp m\bar{3}m\perp m$ of order 48^2 × 2 = 4608. Its cell is defined by three geometrical parameters: the side lengths of the two cubes and the height of the hyperprism.

(iii) The (hypercube 4-dim.) cube crystal family. Its holohedry is group $(m\bar{3}m.[8])\perp m\bar{3}m$ of order 384 × 48 = 18 432. Its cell is defined by two parameters: the side lengths of the hypercube and of the cube.

In space E^8, there exist ten di cubic crystal families. To define them, we use all possible decompositions of space E^8 in orthogonal subspaces:

(i) $E^8 = E^6 + E^2$ or $E^8 = E^6 + E^1 + E^1$. We obtain four families: the (monoclinic di cubes) rectangle, oblique, square and hexagon crystal families. For example, the (monoclinic di cubes) square crystal family holohedry is group $m\bar{3}m(m\bar{3}m)\perp 4mm$ of order 48 × 8 = 384.

(ii) $E^8 = E^3 + E^3 + E^2$ or $E^8 = E^3 + E^3 + E^1 + E^1$. We obtain the four families (di cubes) rectangle, oblique, square and hexagon. For example, the (di cubes) rectangle crystal family holohedry is group $m\bar{3}m\perp m\bar{3}m\perp mm$ of order 48^2 × 4 = 9216.

(iii) $E^8 = E^4 + E^3 + E^1$. We obtain only one crystal family: the (hypercube 4-dim. cube)-al crystal family. The holohedry of this family is group $(m\bar{3}m.[8])\perp m\bar{3}m\perp m$ of order 48 × 8 × 48 × 2 = 36 864.

(iv) $E^8 = E^4 + E^4$. We obtain the di hypercubes 4-dim. crystal family. The family holohedry is group $(m\bar{3}m.[8])\perp(m\bar{3}m.[8])$ of order 384^2 = 147 456.

10. Conclusion

This paper shows that while the computer program is very useful, the geometric method is essential for predicting the crystal families, giving them names and describing their cells, and for assigning the WPV symbols of their holohedries and of all their point groups, therefore enabling the determination of their elements, as we show again for the tri cubic families.

The tri cubic family cells are built from three cubes, or hypercubes, with different side lengths belonging to three disjoint subspaces of space E^n with $n \geq 9$. Their holohedry symbols contain three and only three symbols $m\bar{3}m$.

(i) Tri cubic crystal families of space E^9. The tri cubics family has for holohedry group $m\bar{3}m\perp m\bar{3}m\perp m\bar{3}m$ of order 48^3 = 110 592. Its cell depends on three parameters of length.

(ii) Tri cubic crystal families of space E^{10}. The (monoclinic di cubes cube)-al crystal family has for holohedry group $m\bar{3}m(m\bar{3}m)\perp m\bar{3}m\perp m$ of order 48^2 × 2 = 4608. Its cell depends on four length parameters and one angle parameter. The hypercube 4-dim. di cubes crystal family has for holohedry group $(m\bar{3}m.[8])\perp m\bar{3}m\perp m\bar{3}m$ of order 48 × 8 × 48^2 = 884 736. Its cell depends on three length parameters.

(iii) Tri cubic crystal families of space E^{11}. The tri cubes hexagon crystal family has for holohedry group $m\bar{3}m\perp m\bar{3}m\perp m\bar{3}m\perp 6mm$ of order 48^3 × 12 = 1 327 104. Its cell depends on four length parameters.

(iv) Tri cubic crystal families of space E^{15}. The hypercube 6-dim. hypercube 5-dim. hypercube 4-dim. crystal family has for holohedry the group $(m\bar{3}m.[8].[\bar{5}].[12])\perp(m\bar{3}m.[8].[\bar{5}])\perp m\bar{3}m.[8].[8]$ of order 48^3 × 8^3 × 10^2 × 12 = 67 947 724 800. Its cell depends on three length parameters.

It is easy to generalize to quadri cubic families and so on.

Hence in a space of 2007 dimensions, there exists one 669 times cubic crystal family, *i.e.* the cell is built from 669 cubes belonging to spaces orthogonal two by two and depends on 669 parameters of length, its holohedry has the symbol $(m\bar{3}m\perp)^{669}$ and its order equals 48^{669}.

APPENDIX *A*
The ten point groups listed in §3.3

(*a*) Point group $\bar{4}3m\perp 2$. Group $\bar{4}3m$ is a cubic group of order 24 (space E^3) and 2 is the oblique family holohedry (space E^2).

Therefore group $\bar{4}3m\perp 2$ of order 48 = 24 × 2 belongs to the cube oblique family (space E^5). To obtain the 48 elements of this group, it is sufficient to form the Cartesian product of each element of group $\bar{4}3m$ with the two elements of group 2 (identity, rotation 2_{tu}).

(*b*) Point group $2\bar{\bar{3}}$ is an iso cubic group of type 24′ isomorphic to group $m\bar{3}$. Its full symbol is $\bar{1}/2\,\bar{\bar{3}}$. Its 24 elements are easily obtained from the alternative symbol $23\times\bar{1}_5$.

(*c*) Point group $432\perp mm$. The subgroup 432 acts in space (xyz), the first m acts in space t and the second m in space u; moreover the two axes t and u are orthogonal and orthogonal to space (xyz). The elements of this point group of order 96 = 48 × 2 are the Cartesian products of the 24 elements of the cubic group 432 with the four elements of group mm (identity, rotation π or 2_{tu}, reflections m_t and m_u). Point group $432\perp mm$ belongs to the cube rectangle family because 432 is a cubic group, or more exactly the rotation group of the cube, and group mm is the holohedry of the rectangle family.

(*d*) Point group $m\bar{3}\perp m$ (space E^5). The generator of the reflection m (of the second factor) is either axis t or axis u. The elements of the point group $m\bar{3}\perp m$ of order 48 = 24 × 2 are the Cartesian products of the 24 elements of cubic group $m\bar{3}$ with the two elements of group m (identity, reflection m_t or m_u). Point group $m\bar{3}\perp m$ belongs to the cube rectangle family.

(*e*) Point group (42 3 2).2. This point group of order 48 = 24 × 2 belongs to the cube rectangle crystal family. Its 48 elements are the Cartesian products of the 24 elements of group 42 3 2 [listed in §3.3(*e*)] with the two elements of group 2 (identity, 2_{tu}).

(*f*) Point group $m\bar{3}\perp 4mm$. As stated in §3.3(*f*), this group belongs to the cube square family. To obtain its 192 = 24 × 8 elements, it is sufficient to form the Cartesian product of each element of group $m\bar{3}$ with the eight elements of group $4mm$ [identity, two rotations of order 4 ($4_{tu}^{\pm 1}$), one rotation of order 2 (2_{tu}) and four reflections m (m_t, m_u, $m_{t\pm u}$)].

(*g*) Point group (2 62).44. Group (2 62) is an iso cubic group of order 24 with $23\times\bar{1}_4$ as an alternative symbol. From this alternative symbol it is easy to list all the 96 = 24 × 4 elements of group (2 62).44. However, it is also possible to use the notation 2 62.

(*h*) Point group $\bar{1}_4\,36\,\bar{1}$, full symbol $\bar{4}/\bar{1}_4\,36\,\bar{1}/\bar{1}$ (see §7.1). As this symbol has no symbol ⊥, × or . between the three parts, this group is an iso cubic group of the cube hexagon family because of the double rotation 36. The generators of this group are $3^{\pm 1}_{x+y\;x+z}6^{\pm 1}_{tu}$ for the double rotation, and $\bar{1}_{xytu}$ and $\bar{1}_{x-y\,z\,t-u}$ for the *homotheties*. The 48 elements of this group are identity, four rotations 2, three *homotheties* $\bar{1}_4$, eight double rotations 33, eight double rotations 36, 12 *homotheties* $\bar{1}$ and 12 rotation reflections $4m$.

(*i*) Point group 23.46. This point group of order 144 = 12 × 12 belongs to the cube hexagon family. Indeed 23 is a well known group of the cubic family, while 46 is a cyclic group generated by a double rotation of order 12 through angles $2\pi/4$ and $2\pi/6$. Rotation through an angle $2\pi/6$ belongs to the hexagon family.

(*j*) Point group (42 3 2).3*m*. This group order is 144 = 24 × 6. Group 3*m* of order 6 belongs to the hexagon family in plane (tu). The 144 elements of point group (42 3 2).3*m* are the Cartesian products of the 24 elements of group 42 3 2 [listed in §3.3(*e*)] with the six elements of group 3*m*. For example, the product 42.3 gives the element 46*m* ($4_{xy}2_{zt}3_{tu}$ equals $4_{xy}6_{tu}m_z$) and 42.*m* gives the element $4\bar{1}$ ($4^{+1}_{xy}2_{zt}m_u$ equals $4^{-1}_{xy}\bar{1}_{ztu}$), and so on. The 144 elements of point group (42 3 2).3*m* are as follows: identity, 21 rotations 2, ten rotations 3, six double rotations 32, 16 double rotations 33, 18 double rotations 42, three reflections *m*, 15 *homotheties* $\bar{1}$, 24 elements 3*m*, 12 elements 6*m*, six rotation inversions $4\bar{1}$ and 12 elements 46*m*.

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